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TRANSIENT RESPONSE OF A PLASTICALLY ANISOTROPIC CYLINDER IN PLANE STRAIN†

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Abstract—An anisotropic plastic analysis of a long cylinder subjected to internal impulsive pressure loading is presented. Motion of the thin-walled cylinder is taken as radially symmetric and radial displacement is assumed uniform through the thickness. Resulting transient response formulas are expressed in terms of elementary functions. Comparisons between transient deflections for plastically anisotropic and isotropic materials indicate differences as large as 25 per cent, even when uniaxial material properties are identical along principal directions in the cylinder reference surface.

NOTATION

a	cylinder radius
C	parameter defined in equation (9)
C_p	plate velocity
E	Young's modulus
h	cylinder thickness
m	slope of approximate yield surface
P, R	plastic anisotropy parameters
$p(t)$	applied pressure pulse
t	time
u	nondimensional radial displacement
V_0	initial outward radial velocity
w	outward radial displacement
X	yield strength in rolling direction of textured sheet
Y	yield strength transverse to rolling direction
Z	yield strength normal to sheet
α	ratio of axial to circumferential stress
ε_{1_y}	yield strain in circumferential direction
$\dot{\varepsilon}_i^p$	plastic strain rates
$\dot{\varepsilon}_1$	total strain rate in circumferential direction
κ	nondimensional initial velocity
λ	arbitrary proportionality factor
ν	Poisson's ratio
ρ	mass density
σ_1	circumferential stress
σ_2	axial stress
σ_{1_y}	yield stress in circumferential direction
σ_x	principal stress along rolling direction
σ_y	principal stress transverse to rolling direction
σ_z	principal stress normal to textured sheet
τ	nondimensional time
ϕ	yield function

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1. INTRODUCTION

SEVERAL years ago, Backofen *et al.* [1] introduced the concept of strengthening certain engineering metals through controlled plastic anisotropy. Termed "texture hardening", the concept consists of controlling the crystallographic texture of sheet metal during deformation processing such that the yield stress in the through-thickness direction is larger than the yield stresses along the principal axes of anisotropy in the plane of the sheet. As a result, the ratio of biaxial to uniaxial strength of the plastically anisotropic sheet is significantly larger than that of a sheet constructed of an isotropic von Mises material. This concept is particularly applicable to certain lightweight alloys. Specifically, several investigators [2–4] have shown that various alloys of titanium, such as Ti–5Al–2·5Sn and Ti–6Al–4V, exhibit a potential for significant texture hardening.

The influence of this initial plastic anisotropy on plane stress problems has been a subject receiving considerable recent attention in the literature. Budiansky and Wang [5] present a theoretical study of the Swift cup test and investigate the influence of drawability of sheet metals on the degree of anisotropy between thickness and in-plane directions. Chern and Nemat-Nasser [6] develop a solution for the expansion of a pin-hole in an infinitely extended, plastically anisotropic disk whose thickness is a function of the radial coordinate. Solutions are obtained for both Tresa and von Mises yield functions. Bratt and Adami [7] investigate the influence of initial anisotropy on the reduction of thin-walled tubes and use a linearized yield condition to obtain solutions. Yang [8] solves a class of axisymmetric plane stress problems of plastically transversely isotropic and power law hardening sheet metal materials. Further, the influence of plastic anisotropy or texture hardening on the static burst strength of spherical and cylindrical pressure vessels has recently been investigated [9, 10]. However, all references cited above are restricted to rigid-plastic materials and static pressure loading. Chen has recently developed solutions for annular plates [11] and tubes [12] constructed of elastic–plastic materials. However, the work is restricted to static loading and response. The author is unaware of any work on problems involving plastically anisotropic materials in which the loading is of a dynamic nature and the transient response is sought.

It is the purpose of this paper, then, to evaluate the influence of initial plastic anisotropy on the response of structures undergoing transient motions. Toward this end, a long cylindrical shell subjected to an internal impulsive pressure loading is analyzed. The shell is constructed of an elastic–plastic texture hardened material with a yield condition and flow rule based on the anisotropic plasticity theory developed by Hill [13]. The analysis can be considered a generalization of earlier work which accounted for a von Mises material only [14].

The solution is accomplished by partitioning the entire elastic–plastic transient response of the uniformly expanding cylinder into a series of phases:

1. The elastic phase, before the stress in any fiber of the cross section reaches the texture-hardened yield surface;
2. The first plastic phase, during which the circumferential stress in the cross section moves along the texture-hardened yield surface in stress space;
3. The second plastic phase, during which the stress state remains fixed at an equilibrium point on the yield surface determined from the flow law for the condition of plane strain;

4. The elastic unloading phase, after outward radial motion of the cylinder has ceased and it begins inward radial motion; in this phase, the entire cross section unloads elastically.

The resulting formulas describing the transient elastic–plastic response of the long cylinder are developed in terms of elementary functions. Using the derived formulas, it is shown for one set of parameters that, compared to an isotropic plastic von Mises material, peak transient deflections may be reduced by 25 per cent if long cylinders are constructed of highly texture-hardened (plastically anisotropic) materials.

2. ELASTIC PLASTIC CYLINDER

Assuming radial displacements are uniform through the thickness, and that displacements are small, the equation governing radially symmetric motions of a thin cylindrical shell, Fig. 1, may be written as:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}+\frac{\sigma_1}{\rho a}=\frac{p(t)}{\rho h} \tag{1}$$

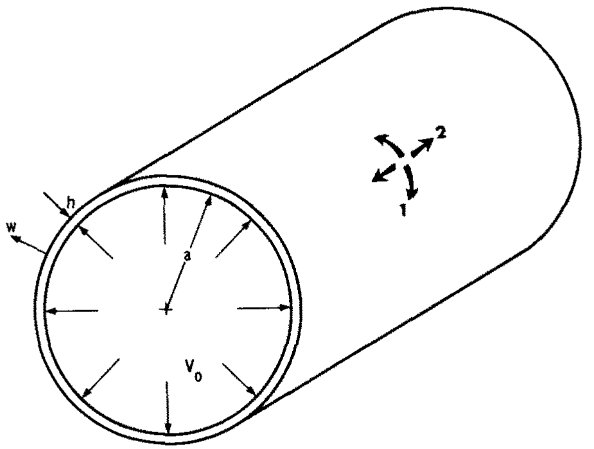

FIG. 1. *Undeformed state of impulsively loaded cylinder.*

where w is outward radial displacement, a is initial cylinder radius, ρ represents mass density, h is thickness, $p(t)$ is the applied pressure pulse and σ_1 denotes the circumferential stress.

Material behavior is assumed to be elastic–perfectly plastic as shown in Fig. 2; the influence of strain hardening and rate sensitivity are ignored. Yielding and plastic flow are governed by a "texture hardened" yield surface, as developed in Appendix A. While the material is plastically anisotropic, it is assumed for simplicity to be isotropic in the elastic region.

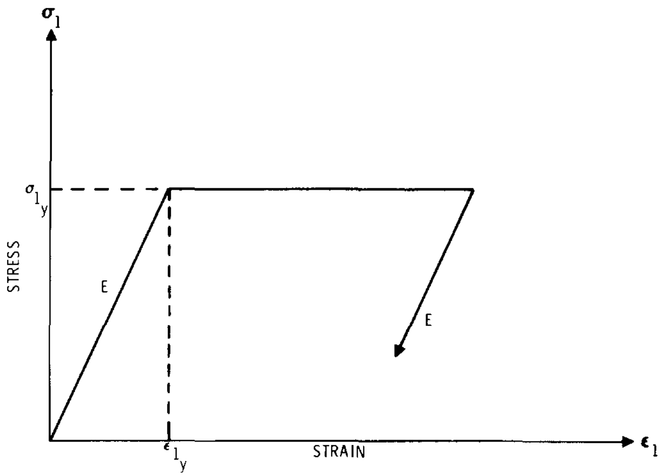

FIG. 2. Elastic-perfectly plastic material behavior along σ_1-direction in stress space.

2.1 *Elastic phase*

The elastic solution with the initial conditions $u(\mathrm{o}) = 0$ and $\mathrm{d}u/\mathrm{d}\tau(\mathrm{o}) = V_o/C_p$ and using Hooke's law is

$$u = \kappa \sin \tau \qquad \tau > 0 \tag{2}$$

where $\kappa = V_o/C_p$, $u = w/a$, $\tau = C_p t/a$, V_o is initial outward radial velocity and $C_p = (E/\rho(1-\nu^2))^{1/2}$, the plate velocity. This elastic solution is valid until the circumferential stress of fibers in the cylinder reaches the yield surface. The cylinder is taken sufficiently thin so that partial yielding does not occur, i.e. the entire cross section yields at the same time.†

The biaxial stress behavior of the cylinder is conveniently represented in the $\sigma_1 - \sigma_2$ plane of principal stress space, as shown in Fig. 3. The circumferential stress during this elastic phase moves from the unstressed point a to point b in Fig. 3, where the anisotropic plastic yield surface is contacted.

2.2 *First plastic phase*

Anisotropic plasticity relations useful to this section are developed in Appendix A in terms of two anisotropy parameters, P and R, since these parameters can be readily determined by direct measurement. As discussed in Appendix A, $P = R = 1$ corresponds to a von Mises material, with no strengthening under balanced biaxial stress; $P = R = 5$ corresponds to a severely texture-hardened material, with significant strengthening under biaxial stress. The yield strength in the through-thickness direction is $\sqrt{3}$ times the strength in the plane of the sheet.

† The time at which yielding occurs is calculated in equation (18).

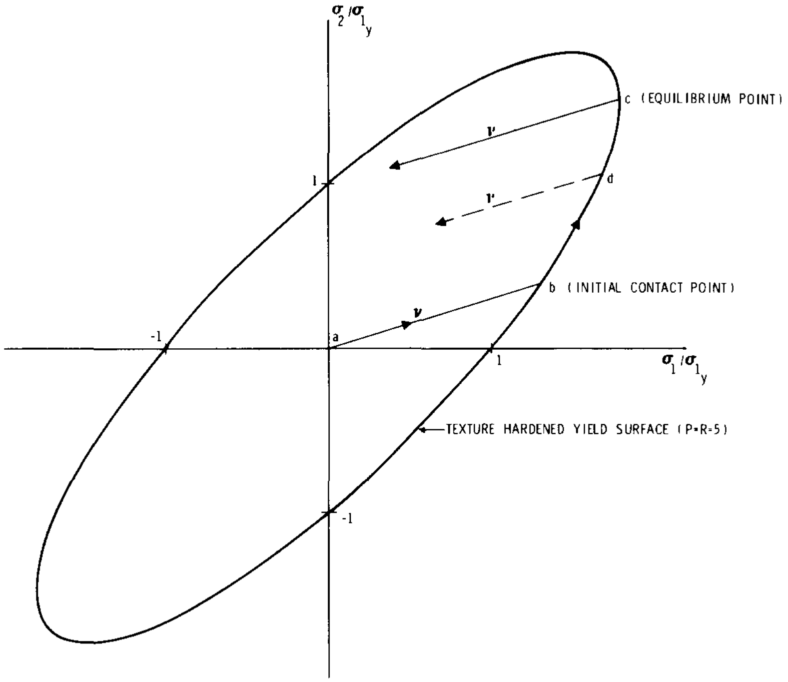

FIG. 3. Locus of the stress state in principal stress space.

There are two possibilities analyzed here:

1. The cylindrical axis is aligned with the rolling direction of the textured sheet. In this case, it is advantageous to utilize the yield condition from equation (A.5) in the form

$$\sigma_x^2+\frac{R(1+P)}{P(1+R)}\sigma_y^2-\sigma_x\sigma_y\left(\frac{2R}{1+R}\right) = X^2 \tag{3}$$

to alleviate scaling difficulties when $P \neq R$. Here, X is the yield strength in the rolling direction. In this case, direction (1) in the cylinder (circumferential direction) is aligned with the x-(rolling) direction of the material and direction (2) with the y-direction.

2. The cylindrical axis is aligned transverse to the rolling direction. In this case, the yield condition is used in the form

$$\frac{P(1+R)}{R(1+P)}\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y\left(\frac{2P}{1+P}\right) = Y^2 \tag{4}$$

where Y is the yield strength in the transverse material direction and where the substitutions of 2 and 1 may be made for x and y, respectively.

In the analysis which follows, situation (1) will be analyzed in detail. Results for case (2) can then be obtained simply by interchanging R and P.

The yield function can be written in terms of shell coordinates as

$$\phi(\sigma_1,\sigma_2) = \sigma_1^2+\frac{R(1+P)}{P(1+R)}\sigma_2^2-\sigma_1\sigma_2\left(\frac{2R}{1+R}\right) = \sigma_{1_y}^2 \tag{5}$$

where σ_{1_y} denotes the yield stress in the circumferential direction. From Hill's anisotropic theory [13], the flow rule is taken of the form

$$\dot{\varepsilon}_i^P = \lambda \frac{\partial \phi}{\partial \sigma_i} \qquad i = 1, 2, 3. \tag{6}$$

The term λ is an arbitrary proportionality factor and $\dot{\varepsilon}_i^P$ denotes plastic strain rates defined for the condition of plane strain, $\dot{\varepsilon}_2 = 0$, as

$$\begin{aligned} \dot{\varepsilon}_1^P &= \dot{\varepsilon}_1 - (\dot{\sigma}_1 - \nu \dot{\sigma}_2)/E \\ \dot{\varepsilon}_2^P &= -(\dot{\sigma}_2 - \nu \dot{\sigma}_1)/E \end{aligned} \tag{7}$$

where the quantities $\dot{\varepsilon}_i$ denote total strain rates. Consider the stress state to be on the texture-hardened yield surface $\phi = 0$ and under a loading condition $\dot{\phi} = 0$. This loading condition can be expressed using the chain rule as

$$\dot{\phi} = 0 = \frac{\partial \phi}{\partial \sigma_1}\dot{\sigma}_1 + \frac{\partial \phi}{\partial \sigma_2}\dot{\sigma}_2$$

or

$$\dot{\sigma}_2 = \dot{\sigma}_1 C \tag{8}$$

where

$$C \equiv -\frac{\partial \phi / \partial \sigma_1}{\partial \phi / \partial \sigma_2}.$$

For the anisotropic yield condition expressed by equation (5), C becomes

$$C = \frac{P(1+R) - RP\alpha}{RP - R(1+P)\alpha} \tag{9}$$

where

$$\alpha = \sigma_2/\sigma_1.$$

α can be written in terms of the anisotropy parameters and the yield stress obtained by manipulation of equation (5) to obtain

$$\alpha = \frac{P}{1+P} - \sqrt{\left\{\left(\frac{P}{1+P}\right)^2 - \frac{P(1+R)}{R(1+P)}\left[1 - \left(\frac{\sigma_{1_y}}{\sigma_1}\right)^2\right]\right\}}. \tag{10}$$

Combining equations (6)–(8), the dynamic state of stress during plastic flow can be written as

$$\begin{aligned} \dot{\sigma}_1 &= \frac{E\dot{\varepsilon}_1}{C^2 - 2C\nu + 1} \\ \dot{\sigma}_2 &= C\dot{\sigma}_1 \end{aligned} \tag{11}$$

where C is obtained from equations (9) and (10).

This nonlinear motion occurs as the stress state, on continued plastic straining, moves from point b (Fig. 3) along the yield surface toward the "zero velocity" point, point c. At this point, hereafter denoted as "equilibrium point", the velocity of the stress state approaches zero (even though there may be continued plastic straining) and the second plastic phase begins. Unfortunately, the incremental equations (11) are nonlinear and it does not appear possible to integrate these relations directly. However, as shown in Fig. 4, the texture hardened yield surface can be reasonably approximated by a straight line† for points lying between point b and the equilibrium point c, at least for values of P and R

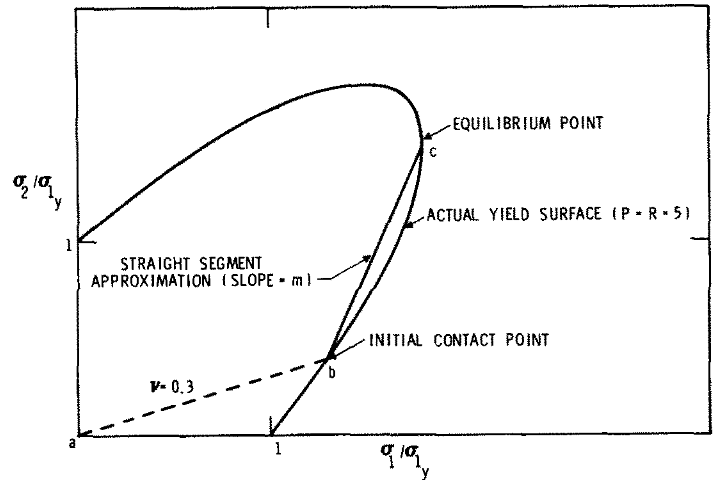

FIG. 4. Texture-hardened yield surface showing straight line approximation.

which are not excessive. The local yield condition can then be expressed as

$$\phi = 0 = m\sigma_1 - \sigma_2 + \frac{(\nu - m)\sigma_{1_y}}{\left[1 - \dfrac{2\nu R}{1+R} + \dfrac{R(1+P)}{P(1+R)}\nu^2\right]^{\frac{1}{2}}} \tag{12}$$

where m is the slope of the line connecting the points of intersection of the radial line of slope ν and the radial line to the "equilibrium point" on the yield surface,

$$m = \frac{\dfrac{P}{P+1}\left(\dfrac{(1+P)(1+R)}{1+P+R}\right)^{\frac{1}{2}}\left(1 - \dfrac{2\nu R}{1+R} + \dfrac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}} - \nu}{\left(\dfrac{(1+P)(1+R)}{1+P+R}\right)^{\frac{1}{2}}\left(1 - \dfrac{2\nu R}{1+R} + \dfrac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}} - 1}. \tag{13}$$

Using the linearized yield condition, equation (12), with equations (6)–(8) it can be shown that the new stress rate expressions become

$$\dot{\sigma}_1 = \frac{E\dot{\varepsilon}_1}{m^2 - 2m\nu + 1} \tag{14}$$

$$\dot{\sigma}_2 = m\dot{\sigma}_1. \tag{15}$$

† A similar linearizing approximation was recently utilized by Bratt and Adami [7].

Equation (14) can be directly integrated, and the following equation of motion can be developed from equation (1):

$$\frac{d^2u}{d\tau^2}+\frac{(1-\nu^2)}{(m^2-2m\nu+1)}u + \frac{\varepsilon_{1_y}(1-\nu^2)(m^2-2m\nu+\nu^2)}{(m^2-2m\nu+1)\left(1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}}} = 0 \tag{16}$$

where ε_{1_y} is the static yield strain in uniaxial tension in the circumferential (transverse) direction.

The initial conditions on the motion during this first plastic phase are

$$u_1 = \kappa \sin\tau_1 = \frac{\varepsilon_{1_y}(1-\nu^2)}{\left(1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}}} \tag{17}$$

$$\frac{du_1}{d\tau} = \kappa\cos\tau_1$$

where τ_1, the time at which the elastic phase ceases, can be determined as

$$\tau_1 = \arcsin\left[\frac{\varepsilon_{1_y}(1-\nu^2)}{\kappa\left[1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right]^{\frac{1}{2}}}\right]. \tag{18}$$

The solution of equations (16) and (17) is given by

$$u = \frac{\varepsilon_{1_y}}{\left[1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right]^{\frac{1}{2}}}[m^2-2m\nu+1]$$
$$\cos\frac{(1-\nu^2)^{\frac{1}{2}}(\tau-\tau_1)}{(m^2-2m\nu+1)^{\frac{1}{2}}}+\frac{\kappa\cos\tau_1(m^2-2m\nu+1)^{\frac{1}{2}}}{(1-\nu^2)^{\frac{1}{2}}} \tag{19}$$
$$\sin\frac{(1-\nu^2)^{\frac{1}{2}}(\tau-\tau_1)}{(m^2-2m\nu+1)^{\frac{1}{2}}}-\frac{\varepsilon_{1_y}(m^2-2m\nu+\nu^2)}{\left[1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right]^{\frac{1}{2}}}.$$

Equation (19) is valid until the circumferential stress in the cylinder reaches the equilibrium point c in Fig. 4.†

† Should outward motion cease before this equilibrium point is reached due to insufficient initial loading, then elastic unloading of the cylinder will occur from some point on the linearized yield surface between b and c. This possibility occurs for

$$B^2-A^2+D^2 < 0$$

where A, B and D are defined after equation (22). A detailed analysis of this possibility is contained in Appendix B.

2.3 *Second plastic phase*

Equation (5) can be used to obtain the circumferential stress at the equilibrium point c (also the point of maximum stress) as

$$(\sigma_1)_{\max} = \sigma_{1_y}\frac{(1+P)(1+R)}{1+P+R}. \tag{20}$$

Using this relation, the deflection (or strain at the cylinder midsurface) can be determined when the stress equilibrium point is just reached as

$$u_2 = \varepsilon_{1_y}\left[(m^2-2m\nu+1)\left(\frac{(1+P)(1+R)}{1+P+R}\right)^{\frac{1}{2}} - \frac{(m^2-2m\nu+\nu^2)}{\left(1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}}}\right] \tag{21}$$

The time at which the equilibrium point is reached can be shown by combining equations (19) and (21) to be

$$\tau_2 = \tau_1 + \frac{(m^2-2m\nu+1)^{\frac{1}{2}}}{(1-\nu^2)^{\frac{1}{2}}} \arcsin\left\{\frac{1}{(D^2+B^2)}[AD - B(B^2-A^2+D^2)^{\frac{1}{2}}]\right\} \tag{22}$$

where

$$A = \varepsilon_{1_y}\left(\frac{(1+P)(1+R)}{1+P+R}\right)^{\frac{1}{2}}(m^2-2m\nu+1)$$

$$B = \frac{\varepsilon_{1_y}}{\left(1-\frac{2\nu R}{1+R}+\frac{R(1+P)}{P(1+R)}\nu^2\right)^{\frac{1}{2}}}(m^2-2m\nu+1)$$

$$D = \frac{\kappa\cos\tau_1(m^2-2m\nu+1)^{\frac{1}{2}}}{(1-\nu^2)^{\frac{1}{2}}}.$$

After time τ_2, the stress state remains fixed at the equilibrium point until elastic unloading occurs. During this subsequent plasticity phase, the circumferential stress is constant and the equation of motion becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\tau^2} + \varepsilon_{1_y}(1-\nu^2)\left[\frac{(1+P)(1+R)}{1+P+R}\right]^{\frac{1}{2}} = 0. \tag{23}$$

The solution [with initial conditions $u(\tau_2) = u_2$ and $\mathrm{d}u/\mathrm{d}\tau(\tau_2) = u_2'$ determined from equations (19) and (22)] is

$$u = u_2 + (\tau-\tau_2)u_2' - \frac{\varepsilon_{1_y}(1-\nu^2)}{2}\left[\frac{(1+P)(1+R)}{1+P+R}\right]^{\frac{1}{2}}(\tau-\tau_2)^2$$

$$\tau > \tau_2. \tag{24}$$

This solution is applicable until outward motion ceases. This time is found from equation (24) as

$$\tau_3 = \tau_2 + \frac{u_2'}{\varepsilon_{1_y}(1-\nu^2)}\left[\frac{1+P+R}{(1+P)(1+R)}\right]^{\frac{1}{2}}. \tag{25}$$

2.4 *Elastic unloading phase*

In the elastic unloading phase, the cylinder unloads elastically from the final state (equilibrium point) on the yield surface along a line of slope ν. Inward motion is governed by

$$\frac{d^2u}{d\tau^2}+u+\varepsilon_{1_y}(1-\nu^2)\left[\frac{(1+P)(1+R)}{1+P+R}\right]^{\frac{1}{2}}-u_3 = 0. \tag{26}$$

The solution [with initial conditions $u(\tau_3) = u_3$ and $du/d\tau(\tau_3) = 0$ determined from equations (24) and (25)] is given by

$$u = \varepsilon_{1_y}(1-\nu^2)\left[\frac{(1+P)(1+R)}{1+P+R}\right]^{\frac{1}{2}}(\cos(\tau-\tau_3)-1)+u_3 \tag{27}$$

$$\tau > \tau_3.$$

This solution is applicable until the stress state contacts the yield surface again.

3. DISCUSSION

To summarize, the transient response of the uniformly expanding long cylinder has been divided into the following phases.

1. The initial elastic phase, before any fibers in the cross section have reached the yield surface. During this phase, the stress state at some location in the cross section proceeds from the origin of stress space (point *a*, Fig. 4) along *a–b* until the yield surface is contacted at point *b*.
2. The fully plastic phase, during which the stress state moves along the yield surface in some nonlinear manner (here a straight line linearizing approximation was taken to obtain a solution). The motion of the stress state in this phase is between points *b* and *c* of Fig. 4. Should outward motion of the cylinder cease before point *c* is reached, elastic unloading will occur, say from some point *d*, along a line with slope equal to Poisson's ratio. However, if point *c* is reached, the stress state remains there during a second fully plastic phase until outward cylinder motion has stopped.
3. The elastic unloading phase, in which the stress state at some location in the cross section unloads elastically from point *c* of Fig. 4 along a line of slope ν and inward motion begins. Solutions presented in the preceding section are valid until the yield surface is subsequently contacted.

The influence of texture hardening on transient cylinder response is shown in Fig. 5 for the set of parameters indicated on the figure. It is seen that the influence of the anisotropy parameter (for $P = R$) is indeed significant. In particular, peak deflection for the (plastically) "transversely isotropic" situation represented by $P = R = 5$† is reduced by more than 25 per cent compared to the isotropic von Mises case ($P = R = 1$) for identical uniaxial yield stresses.

It is also shown in Fig. 5 that differences in response between rings (uniaxial stress) and long cylinders (plane strain) for the same uniaxial yield stress become even more pronounced than reported earlier [14] as the degree of plastic anisotropy is increased.

† This degree of texture hardening was chosen as a reasonable upper limit for purposes of comparison. Anisotropy values of this magnitude have been reported by several investigators [1, 3].

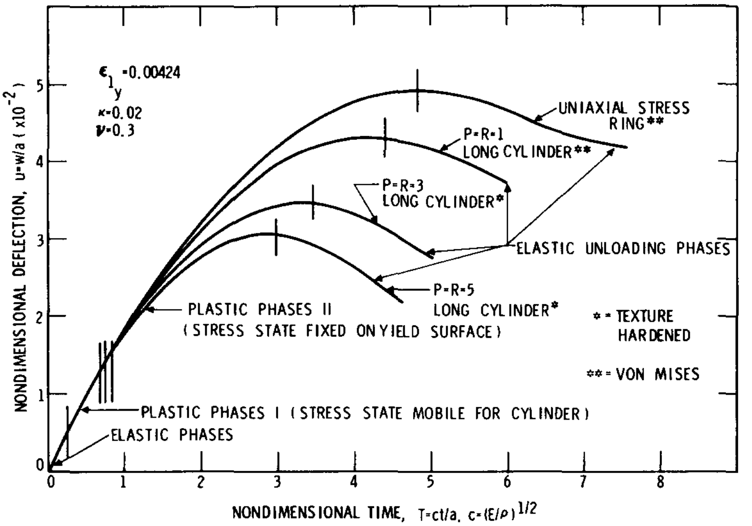

FIG. 5. Influence of texture hardening on transient cylinder response.

Transient deflections are plotted in Fig. 6 for a highly texture-hardened material. Deflections are plotted for two values of Poisson's ratio. It is seen that deflections are somewhat dependent on Poisson's ratio, at least for large values of the anisotropy parameters.

4. CONCLUSIONS

In this paper, the influence of plastically anisotropic material behavior on the transient response of a long cylindrical shell subjected to impulsive loading has been investigated.

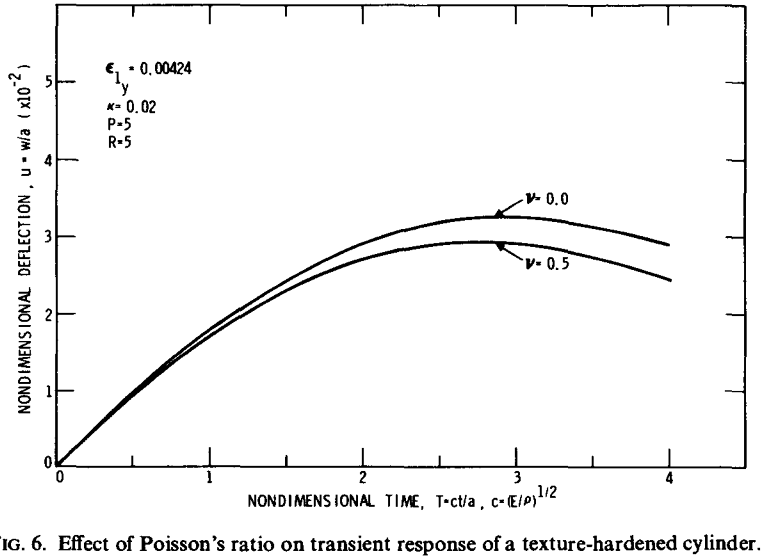

FIG. 6. Effect of Poisson's ratio on transient response of a texture-hardened cylinder.

Formulas for the transient radial deflections of the cylinder are developed in terms of elementary functions, and the following conclusions are drawn from numerical examples based on these formulas:

1. The biaxial strengthening of materials exhibiting extreme "texture hardening" behavior is found to reduce peak deflections of a long, uniformly loaded tube by 25 per cent compared to isotropically plastic response.

2. Differences in response between rings (uniaxial stress) and long cylinders (plane strain) become more pronounced as the degree of plastic anisotropy is increased.

3. Cylinder deflections are found to be somewhat dependent upon Poisson's ratio for large values of the plastic anisotropy parameters.

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APPENDIX A

Most of the resulting anisotropic plastic relations developed in this appendix have been presented by others. The derivations are presented here for completeness.

Hill's [13] anisotropic yield criterion can be written for a state of plane stress ($\sigma_z = 0$) and in terms of principal stresses as

$$\frac{\sigma_x^2}{X^2}+\frac{\sigma_y^2}{Y^2}-\left[\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right]\sigma_x\sigma_y = 1 \tag{A.1}$$

where X, Y, Z denote, respectively, the uniaxial yield stresses along the Cartesian x, y and z axes, the principal axes of anisotropy. It is assumed that the x- and y-axes lie in the plane of a thin sheet of texture-hardened metal and that the z-axis is normal to the plane of the sheet. The x-axis is taken to be parallel to the rolling direction and the y-axis transverse to the rolling direction.

Since direct measurement of Z is difficult for thin sheets, it is convenient to rewrite equation (A.1) in terms of material constants which can be determined by direct measurement. Two such anisotropy parameters are P and R, defined as follows:

$$\begin{aligned} P &= \mathrm{d}\varepsilon_x/\mathrm{d}\varepsilon_z \Big|_{\sigma_x = \sigma_z = 0} \\ R &= \mathrm{d}\varepsilon_y/\mathrm{d}\varepsilon_z \Big|_{\sigma_y = \sigma_z = 0.} \end{aligned} \tag{A.2}$$

The parameters P and R are determined by measuring the incremental width-to-thickness strains in coupons subjected, respectively, to uniaxial stress along the y- and x-directions.

From Hill [13] the stress–strain increment equations, determined by applying the flow rule to the yield condition, can be reduced to

$$\begin{aligned} \mathrm{d}\varepsilon_x &= \mathrm{d}\lambda\left[\frac{\sigma_x}{X^2} - \frac{\sigma_y}{2}\left\{\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right\}\right] \\ \mathrm{d}\varepsilon_y &= \mathrm{d}\lambda\left[\frac{\sigma_y}{Y^2} - \frac{\sigma_x}{2}\left\{\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right\}\right] \\ \mathrm{d}\varepsilon_z &= \mathrm{d}\lambda\left[-\frac{\sigma_x}{2}\left\{\frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2}\right\} - \frac{\sigma_y}{2}\left\{\frac{1}{Y^2}+\frac{1}{Z^2}-\frac{1}{X^2}\right\}\right]. \end{aligned} \tag{A.3}$$

Equations (A.2) and (A.3) can be combined to obtain

$$P = \frac{\left\{\dfrac{1}{X^2}+\dfrac{1}{Y^2}-\dfrac{1}{Z^2}\right\}}{\left\{\dfrac{1}{Y^2}+\dfrac{1}{Z^2}-\dfrac{1}{X^2}\right\}} \tag{A.4}$$

and

$$R = \frac{\left\{\dfrac{1}{X^2}+\dfrac{1}{Y^2}-\dfrac{1}{Z^2}\right\}}{\left\{\dfrac{1}{Z^2}+\dfrac{1}{X^2}-\dfrac{1}{Y^2}\right\}}.$$

Using equation (A.4), equation (A.1) can be written in two equivalent expressions, both of which are useful in the analysis in the elastic–plastic cylinder section:

$$\begin{aligned} \frac{P(1+R)}{R(1+P)}\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y\left(\frac{2P}{1+P}\right) &= Y^2 \\ \sigma_x^2 + \frac{R(1+P)}{P(1+R)}\sigma_y^2 - \sigma_x\sigma_y\left(\frac{2R}{1+R}\right) &= X^2. \end{aligned} \tag{A.5}$$

The influence of the anisotropy parameters, P and R, on the yield condition is indicated in Fig. 7. It is seen that for $P = R$, corresponding to a "transversely isotropic" material in which $X = Y$, the yield surface undergoes elongation under biaxial stress and retains symmetry about the dashed $\sigma_x = \sigma_y$ line. $P = R = 1$ corresponds to the isotropic von Mises

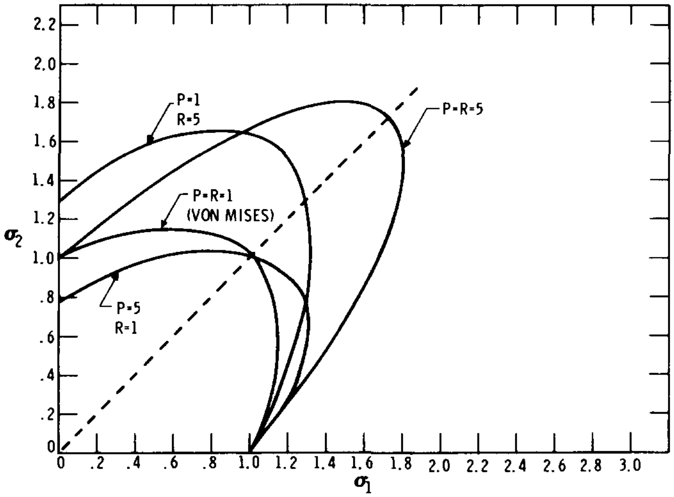

FIG. 7. Influence of anisotropy parameters on yield surface.

ellipse, while larger equal values of P and R indicate increased texture hardening. It is seen that for balanced biaxial tension, the yield strength for $P = R = 5$ is nearly double that of the isotropic von Mises situation which exhibits no biaxial strengthening for balanced biaxial tension.

The situation $P \neq R$ (orthotropic material) is also included in Fig. 7 where it can be seen that unequal values of P and R correspond to a rotation of the yield surface away from the $\sigma_x = \sigma_y$ line of symmetry.

APPENDIX B

The possibility of elastic unloading before reaching the equilibrium point c in Fig. 4 is considered here.

The time at which outward motion ceases can be found from equation (19) as

$$\tau_2 = \tau_1 + \frac{(m^2 - 2mv + 1)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}} \arctan\left[\frac{\kappa\left\{1 - \frac{2vR}{1+R} + \frac{R(1+P)}{P(1+R)}v^2\right\}^{\frac{1}{2}} \cos\tau_1}{\varepsilon_{1_y}(1-v^2)^{\frac{1}{2}}[m^2 - 2mv + 1]^{\frac{1}{2}}}\right]. \tag{B.1}$$

At this time, the cylinder begins inward motion and unloads elastically along a line of slope v. From Hooke's law and equation (1), the equation of motion becomes

$$\frac{d^2u}{d\tau^2} + u + \left(u_2 - \frac{\varepsilon_{1_y}(1-v^2)}{\left\{1 - \frac{2vR}{1+R} + \frac{R(1+P)}{P(1+R)}v^2\right\}^{\frac{1}{2}}}\right)\left(\frac{(1-v^2)}{(m^2 - 2mv + 1)} - 1\right) = 0 \tag{B.2}$$

where u_2 is given by equation (19) for $\tau = \tau_2$. The solution with initial conditions $u(\tau_2) = u_2$ and $du/d\tau(\tau_2) = 0$ is

$$u = \left[u_2 + \left(u_2 - \frac{\varepsilon_{1y}(1-\nu^2)}{\left\{1 - \dfrac{2\nu R}{1+R} + \dfrac{R(1+P)}{P(1+R)}\nu^2\right\}^{\frac{1}{2}}}\right)\left(\frac{(1-\nu^2)}{(m^2-2m\nu+1)} - 1\right)\right]\cos(\tau - \tau_2)$$
$$+ \left(u_2 - \frac{\varepsilon_{1y}(1-\nu^2)}{\left\{1 - \dfrac{2\nu R}{1+R} + \dfrac{R(1+P)}{P(1+R)}\nu^2\right\}^{\frac{1}{2}}}\right)\left(1 - \frac{(1-\nu^2)}{(m^2-2m\nu+1)}\right) \qquad \tau > \tau_2. \tag{B.3}$$

This solution is applicable until the stress state contacts the yield surface again in either second or third quadrant of principal stress space.

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Абстракт—В области анизотропной теории пластичности дается анализ для длинного цилиндра, подверженного действию импульсивной внутренней нагрузки давления. Учитивается движение тонкостенного цилиндра как радиально-сим-метрическое. Предлолагается радиальное перемещение постоянное сквозь толщину. Определяются суммарные формулы переходных характеристик в виде элементарных функцмй. Сравнения между переходными прогибами для пластически анизотропных и изотропных материалов указывают разницм выше 25%, даже для случая когда соосные свойства материала одинаковые вдоля главных направлений, в относительной поверхности цилиндра.